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THE STABILITY OF A PARTICULAR MOTION OF A SOLID WITH A VISCOELASTIC MEMBRANE IN A CIRCULAR ORBIT†

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A system consists of a bearing solid and a circular viscoelastic membrane attached to the solid along its contour. The system in the undeformed state is dynamically symmetrical about an axis orthogonal to the plane of the membrane. The motion of this system on a circular orbit in a central Newtonian gravitational field is investigated within the limits of linear elasticity theory. Quasistatic motions of the system are considered, on the assumption that the membrane is sufficiently stiff and the dissipative forces are small compared with the elastic forces. A particular motion is found in which the plane of the membrane lies parallel to the orbital plane and the system revolves on its axis of symmetry, at a fixed angular velocity of arbitrary magnitude. The stability of this motion is examined. It is shown that, compared with the parallel results for a symmetrical satellite–solid system, the presence of the viscoelastic membrane makes the stability regions smaller and implies the existence of asymptotic stability with respect to part of the variables.

1. LET US consider a system consisting of a bearing solid and a circular viscoelastic membrane attached to the solid along its contour. The axis of symmetry of the membrane is one of the principal central axes of inertia of the system in the undeformed state and is also an axis of dynamical symmetry for the system.

We shall assume that the system is moving in a central Newtonian gravitational field in a circular orbit and that its motion around its centre of mass does not affect the motion of the centre of mass itself.

Let $Ox_1y_1z_1$ and $Gxyz$ be two coordinate frames, with their origins at the centre of mass O of the undeformed system and the centre of mass G of the deformed system, respectively; the axes are directed along and parallel to the principal central axes of inertia in the undeformed state. The z_1 axis coincides with the axis of symmetry of the membrane.

Treatment of the system (solid plus membrane) using linear elasticity theory yields [1] a system of equations that is valid for any inertia tensor. Those of the equations that describe the motion of the system as a whole about its centre of mass (the equations of motion of the trihedron $Gxyz$), assuming dynamical symmetry, are as follows (throughout, summation is performed from $m = 1$ to $m = \infty$):

$$\begin{aligned}
&[A\omega_1^{\cdot} + (C - A)\omega_2\omega_3 - 3\omega_0^2 (C - A)\gamma_2\gamma_3] + 2l \textstyle\sum b_m q_{m0}' \times \\
&\quad \times [\omega_1^{\cdot} - \omega_2\omega_3 + 3\omega_0^2\gamma_2\gamma_3] + 2l\omega_1 \textstyle\sum b_m q_{m0}^{\cdot\prime} + \\
&+ \textstyle\sum a_m q_{m1}' [-\omega_3^{\cdot} - \omega_1\omega_2 + 3\omega_0^2\gamma_1\gamma_2] + \textstyle\sum a_m q_{m1}^{\cdot\cdot\prime\prime} + \\
&+ \textstyle\sum a_m q_{m1}'' [(\omega_3^2 - \omega_2^2) - 3\omega_0^2 (\gamma_3^2 - \gamma_2^2)] = 0
\end{aligned} \tag{1.1}$$

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$$[A\omega_2^{\cdot} - (C - A)\omega_1\omega_3 + 3\omega_0^2 (C - A)\gamma_1\gamma_3] + 2l \sum b_m q_{m0}{}' \times$$
$$\times [\omega_2^{\cdot} + \omega_1\omega_3 - 3\omega_0^2\gamma_1\gamma_3] + 2l\omega_2 \sum b_m q_{m0}{}^{\cdot\prime} -$$
$$- \sum a_m q_{m1}{}^{\cdot\cdot\prime} + \sum a_m q_{m1}{}' [(\omega_1^2 - \omega_3^2) - 3\omega_0^2 (\gamma_1^2 - \gamma_3^2)] +$$
$$+ \sum a_m q_{m1}{}'' [-\omega_3^{\cdot} + \omega_1\omega_2 - 3\omega_0^2\gamma_1\gamma_2] = 0 \quad (1.2)$$
$$C\omega_3^{\cdot} - \sum a_m q_{m1}{}' [\omega_1^{\cdot} - \omega_2\omega_3 + 3\omega_0^2\gamma_2\gamma_3] - 2\omega_1 \sum a_m q_{m1}{}^{\cdot\prime} -$$
$$- \sum a_m q_{m1}{}'' [\omega_2^{\cdot} + \omega_1\omega_3 - 3\omega_0^2\gamma_1\gamma_3] - 2\omega_2 \sum a_m q_{m1}{}^{\cdot\prime\prime} = 0 \quad (1.3)$$

Here A and C are the moments of inertia of the undeformed system relative to the axes O_{x1} and Oz_1; ω_1, ω_2, ω_3 and γ_1, γ_2, γ_3 are the projections on the Gx, Gy, Gz axes of the absolute angular velocity ω of the trihedron $Gxyz$ and the unit vector γ along the radius-vector of the centre of mass G relative to the attracting centre; ω_0 is the mean motion of the centre of mass on the orbit. The generalized coordinates $q_{m0}{}'$, $q_{m1}{}'$, $q_{m1}{}''$ $(m = 1, 2, \ldots)$ were described in [1]. The coefficients a_m and b_m are evaluated from the formulas

$$a_m = \pi\sigma c_{m1} \int_0^a J_1(k_{m1}\rho)\,\rho^2\,d\rho, \quad b_m = 2\pi c_{m0}\sigma \int_0^a J_0(k_{m0}\rho)\,\rho\,d\rho \quad (m = 1, 2, \ldots)$$

where a is the radius of the membrane and σ is its surface density, $J_n(k_{mn}\rho)$ is the Bessel function of nth order, the parameter k_{mn} is the mth root of the equation $J_n(ka) = 0$, and the quantities c_{mn} are defined by $c_{mn} = (\pi\sigma a_2 J_n{}'^2(k_{mn}a)/2)^{-1/2}$ $(n = 0, 1;\ m = 1, 2, \ldots)$. The letter l in (1.1) and (1.2) denotes the distance from the centre of mass O to the centre of the undeformed membrane.

2. We will consider quasistatic motions of the system [2, 3], in which the elastic vibrations of the membrane are forced ones excited by the gravitational and inertial forces. Let us assume that the characteristic damping time of the free elastic vibrations of the membrane is much greater than the characteristic period of the elastic vibrations, but much less than the period T_0 of one revolution of the centre of mass on the orbit. Under these assumptions, the values of the generalized coordinates $q_{m0}{}^{\circ}$, $q_{m1}{}'$, $q_{m1}{}''$ $(m = 1, 2, \ldots)$ are calculated by the following formulas [1]:

$$q_{mn} = \frac{\varepsilon^2}{\lambda_{mn}^2}[Q_{mn} - 2\chi b \dot{Q}_{mn}] + O(\varepsilon^4) \quad (n = 0, 1;\ m = 1, 2, \ldots)$$
$$Q_{m0}{}' = b_m l [\omega_1^2 + \omega_2^2 - \omega_0^2 (1 - 3\gamma_3^2)]$$
$$Q_{m1}{}' = a_m [\omega_2^{\cdot} - \omega_1\omega_3 + 3\omega_0^2\gamma_1\gamma_3] \quad (2.1)$$
$$Q_{m1}{}'' = -a_m [\omega_1^{\cdot} + \omega_2\omega_3 - 3\omega_0^2\gamma_2\gamma_3]$$

Here $\varepsilon = \omega_0/\Omega_1$ is a small parameter (we have assumed that $T_0 \sim 1$), Ω_1 is the least natural frequency of elastic vibrations of the membrane and χ is a dimensionless parameter and b is the positive constant appearing in the Rayleigh dissipative function [1]. By virtue of our assumptions, the parameters ε and χ satisfy the inequalities $0 < \chi \ll \varepsilon \ll 1$; in deriving (2.1) we assumed that $\chi \sim \varepsilon^{\delta+1}(0 < \delta < 1)$. The quantity λ_{mn} in (2.1) is given by $\lambda_{mn} = \varepsilon^{-1}\omega_{mn}$, where ω_{mn} is the corresponding natural frequency of elastic vibrations of the membrane; the latter is related to the parameter k_{mn} by $\omega_{mn} = ak_{mn}$ $(n = 0, 1;\ m = 1, 2, \ldots)$.

3. We will now introduce an orbital coordinate frame $GXYZ$, whose axes GX, GY and GZ are directed, respectively, along the transversal to the orbit, the binormal and the radius-vector of the centre of mass G relative to the attracting centre. The orientation of the frame $Gxyz$ relative to $GXYZ$ is specified in terms of Euler angles ψ, θ, φ.

We put $\omega_1 = \omega_0 p$, $\omega_2 = \omega_0 q$, $\omega_3 = \omega_0\beta$ in Eqs (1.1)–(1.3), introduce a parameter $\alpha = C/A$ and transform to a new independent variable $\tau = \omega_0 t$. Using the kinematic relations (the dot, as before, denotes differentiation with respect to the independent variable)

$$\gamma_1 = \sin\varphi\sin\theta, \quad \gamma_2 = \cos\varphi\sin\theta, \quad \gamma_3 = \cos\theta$$
$$p = \psi^{\cdot}\sin\theta\sin\varphi + \theta^{\cdot}\cos\varphi + \sin\psi\cos\varphi + \cos\psi\sin\varphi\cos\theta$$
$$q = \psi^{\cdot}\sin\theta\cos\varphi - \theta^{\cdot}\sin\varphi - \sin\psi\sin\varphi + \cos\psi\cos\varphi\cos\theta \quad (3.1)$$
$$\beta = \psi^{\cdot}\cos\theta + \varphi^{\cdot} - \cos\psi\sin\theta$$

we can write Eqs (1.1)–(1.3) in terms of the Euler angles. Multiplying the first of the resulting equations by $\sin\varphi$, the second by $\cos\varphi$ and adding, then multiplying the first equation by $\cos\varphi$, the second by $-\sin\varphi$ and adding, and retaining the third equation unchanged, we obtain the following system of equations:

$$\begin{gathered}
(1 + P_0)[\psi^{\cdot\cdot} \sin\theta + 2\psi^{\cdot}\theta^{\cdot} \cos\theta - \theta^{\cdot} l_4 - l_3 \sin\psi] - \\
- (\alpha - 1 - P_0)\beta l_1 + P_0^{\cdot} l_2 - P_2^{**} + P_1(-\beta^{\cdot} + l_1 l_2) + \\
+ P_2(-\beta^2 + 3\cos^2\theta + l_1^2) = 0 \\
(1 + P_0)[\theta^{\cdot\cdot} - \psi^{\cdot 2} \sin\theta\cos\theta + \psi^{\cdot} l_4 \sin\theta + l_3 \cos\psi\cos\theta] + \\
+ (\alpha - 1 - P_0)(\beta l_2 - 3\sin\theta\cos\theta) + P_0^{\cdot} l_1 + P_1^{**} + \\
+ P_1(\beta^2 - 3\cos 2\theta - l_2^2) - P_2(\beta^{\cdot} + l_1 l_2) = 0 \qquad (3.2) \\
\beta^{\cdot} + P_2(\beta l_2 - 3\sin\theta\cos\theta) - P_1 l_1 - 2(P_1^* l_2 + P_2^* l_1)/\alpha = 0 \\
l_1 = \theta^{\cdot} + \sin\psi, \quad l_2 = \psi^{\cdot}\sin\theta + \cos\psi\cos\theta, \quad l_3 = \beta + \cos\psi\sin\theta \\
l_4 = l_3 + \cos\psi\sin\theta, \quad P_0 = 2l\sum b_m q_{m0}'/A \\
P_1 = \sum (a_m q_{m1}' \sin\varphi + q_{m1}'' \cos\varphi)/A, \quad P_2 = \sum (a_m q_{m1}' \cos\varphi - q_{m1}'' \sin\varphi)/A \\
P_1^* = \sum (a_m q_{m1}^{\cdot\prime} \sin\varphi + q_{m1}^{\cdot\prime\prime} \cos\varphi)/A, \quad P_2^* = \sum (a_m q_{m1}^{\cdot\prime} \cos\varphi - \\
- q_{m1}^{\cdot\prime\prime} \sin\varphi)/A \\
P_1^{**} = \sum (a_m q_{m1}^{\cdot\cdot\prime} \sin\varphi + q_{m1}^{\cdot\cdot\prime\prime} \cos\varphi)/A, \quad P_2^{**} = \sum (a_m q_{m1}^{\cdot\cdot\prime} \cos\varphi - \\
- q_{m1}^{\cdot\cdot\prime\prime} \sin\varphi)/A
\end{gathered}$$

Inserting the values of the generalized coordinates q_{m0}', q_{mi}', q_{mi}'' ($i = 1, 2, \ldots$) from (2.1) and the corresponding values of their derivatives, we obtain equations describing the motion of the "body plus membrane" system as a whole relative to its centre of mass. Henceforth we shall ignore quantities of the order of ε^4 and higher in these equations.

Using the expressions

$$\begin{gathered}
Q_{m0}' = b_m l \omega_0^2 [\psi^{\cdot 2} \sin^2\theta + \theta^{\cdot 2} + 2\psi^{\cdot} \sin\theta\cos\theta\cos\psi + 2\theta^{\cdot} \sin\psi - \\
- \cos^2\psi \sin^2\theta + 3\cos^2\theta] \\
\dot{q}_{m1} \sin\varphi + \ddot{q}_{m1} \cos\varphi = \frac{a_m \omega_0^2 (2-\alpha)\varepsilon^2}{\lambda_{m1}^2} \{(3\sin\theta\cos\theta - \beta l_2) + \\
+ 2\omega_0 \chi b [\beta^2(\alpha - 1) l_1 - 3\theta^{\cdot} \cos 2\theta]\} \qquad (3.3) \\
\dot{q}_{m1} \cos\varphi - \ddot{q}_{m1} \sin\varphi = \frac{a_m \omega_0^2 (2-\alpha)\varepsilon^2}{\lambda_{m1}^2} \{-\beta l_1 + 2\omega_0 \chi b [3\beta(\alpha - 2)\sin\theta\cos\theta - \\
- \beta^2(\alpha - 1) l_2 + 3\sin\theta\cos\theta(\psi^{\cdot}\cos\theta - \cos\psi\sin\theta)]\}
\end{gathered}$$

which are obtained from (2.1), (3.1) and Eqs (3.2) (after the above-mentioned substitution) by setting $\varepsilon = 0$, and by differentiating one can show that the left-hand sides of Eqs (3.2) do not involve the angle φ.

4. It follows from (2.1), (3.2) and (3.3) that the equations of motion of the system as a whole admit of a particular solution in quasistatic motion:

$$\psi = \pi, \quad \theta = \pi/2, \quad \beta = \beta_0 = \text{const} \qquad (4.1)$$

corresponding to motion of the system when the plane of the membrane is parallel to the orbital plane, and the system revolves uniformly on the axis of symmetry of the membrane at an angular velocity of arbitrary magnitude.

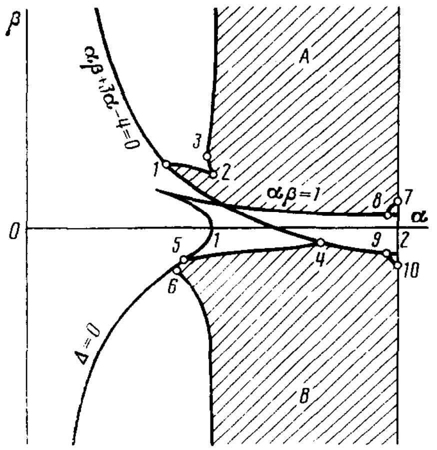

FIG. 1.

We will now investigate the stability of this particular solution with respect to perturbations of ψ, θ, $\psi^{\bullet}$, $\theta^{\bullet}$, β, setting $\psi=\pi+x_1$, $\theta=\pi/2+x_2$, $\beta=\beta_0+x_3$. The characteristic equation of the linearized system of equations (3.2) obtained by (3.3) may be written as

$$\lambda\,(a_0\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4)=0 \tag{4.2}$$

$$a_0=1+O(\varepsilon^2),\quad a_2=\alpha^2\beta^2-2\alpha\beta+3\alpha-1+O(\varepsilon^2)$$

$$a_4=(\alpha\beta-1)(\alpha\beta+3\alpha-4)+O(\varepsilon^2)$$

$$a_1=Lb_1,\quad a_3=Lb_3,\quad L=2\varkappa_1 b\omega_0^3\,(2-\alpha)\varepsilon^2\chi/A>0$$

$$b_1=2\,(\alpha-1)\beta^4+3\,(2-\alpha)[(\alpha+1)\beta^2-2\beta+3]$$

$$b_3=2\,(\alpha+2)\beta^4-6\,(4-\alpha)\beta^3+3\,(3-\alpha)(\alpha+2)\beta^2+3\,(2-\alpha)\times$$

$$\times\,(3\alpha-2)\beta-9\,(2-\alpha),\quad \varkappa_1=\sum a_m^2/\lambda_{m1}^2$$

This equation has one zero root and two pairs of complex-conjugate roots.

In the case of the symmetric satellite—a rigid body ($\varepsilon=0$), the plane of the parameters (α,β) contains two regions (see Fig. 1) A and B in which the complex roots of Eq. (4.2) are purely imaginary. These regions are bounded by the curves $\alpha\beta=1$, $\alpha\beta+3\alpha-4=0$, $\Delta=(\alpha^2\beta^2-2\alpha\beta+3\alpha-1)^2-4(\alpha\beta-1)(\alpha\beta+3\alpha-4)=0$ represented in Fig. 1 by solid lines, defined by the following systems of inequalities: $\alpha\beta>1$, $\alpha\beta+3\alpha-4>0$, $\Delta>0$ for A and $\alpha\beta<1$, $\alpha\beta+3\alpha-4<0$, $\Delta>0$ for B. As shown by investigations of the corresponding non-linear problem [4–6], the above particular solution is stable in the region A; it is also stable in B everywhere except along two segments of the fourth-order resonance curve and possibly one additional point of the region. Outside the regions A and B the characteristic equation (4.2) (with $\varepsilon=0$) has a pair of roots with positive real parts and the solution (4.1) is unstable.

At small but non-zero values of ε, Eq. (4.2) will again have a pair of roots with positive real parts outside A and B, and the solution (4.1) is unstable. We will now consider the stability question in the regions A and B.

Analysing the structure of equations (3.2) with due attention to (2.1), (3.3), one can show that the critical case occurring here is singular [7], so that the Lyapunov–Malkin theorem is applicable. If the conditions of the Routh–Hurwitz criterion

$$b_1>0,\quad b_3>0,$$

$$b_3\,(b_1a_2-a_0b_3)-a_1b_1^2>0 \tag{4.3}$$

are satisfied, the real parts of the complex roots of Eq. (4.2) are negative and the solution (4.1) will be stable with respect to the variables ψ, θ, $\psi^{\bullet}$, $\theta^{\bullet}$, β, for sufficiently small ε; it will be asymptotically stable with respect to the variables ψ, θ, $\psi^{\bullet}$, $\theta^{\bullet}$.

Computer investigation of inequalities (4.3) in the regions A and B yielded the stability regions shown hatched in the figure.

The curve in A that separates the stability and instability regions intersects the hyperbola $\alpha\beta + 3\alpha - 4 = 0$ at the point 1 with coordinates (0.7335, 2.4533) and has a cusp 2 (1, 2), a vertical tangent at the point 3 (0.9714, 2.4457) and a vertical asymptote $\alpha = 1$ as $\beta \to +\infty$. The curve bounding the stability region in B also has a vertical asymptote $\alpha = 1$ as $\beta \to -\infty$; it intersects the hyperbola $\alpha\beta + 3\alpha - 4 = 0$ at the point 4 (1.5846, −0.4757), touches the curve $\Delta = 0$ at the point 5 (0.8235, −1.2313) and has a vertical tangent at the point 6 (0.8047, −1.439). Near the boundary $\alpha = 2$ of the regions A and B there are small regions of instability in a neighbourhood of the curves $\alpha\beta = 1$ and $\alpha\beta + 3\alpha - 4 = 0$. The coordinates of their characteristic points, shown in the figure, are as follows: 7 (2, 1); 8 (1.9529, 0.5121); 9 (1.9737, −0.9733); 10 (2, −1.1083).

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